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# Exact isolated solutions for the class of quantum optical systems

Marek Kuś† and Maciej Lewenstein‡

† Institute of Theoretical Physics, University of Warsaw 00-681 Warsaw, Hoża 69, Poland
‡ Institute for Theoretical Physics, Polish Academy of Sciences 00-668, Aleja Lotników 32/46, Poland

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**Abstract.** We present a method of obtaining exact isolated solutions for the class of quantum optical systems without the use of a rotating wave approximation (RWA). The method generalises the results known from the literature to the case of multilevel atomic systems. The analytical properties of the solutions in the Bargmann representation for the radiation field mode are discussed. The analogues of these solutions for atomic systems interacting with an external field are constructed.

## 1. Introduction

Optical non-integrable systems in which the rotating wave approximation (RWA) is not employed have enjoyed a growing popularity in recent years. The statement above relates especially to the simplest model of the two-level atom interacting with one mode of the electromagnetic field (i.e. a spin-$\frac{1}{2}$ coupled to a harmonic oscillator). This fact has manifold reasons. Firstly the recent progress in the Rydberg atoms' technique allows for the experimental studies of interaction of atoms with a single cavity mode (Haroche 1983). Due to large values of Rydberg atoms' dipole moments and a cavity mode wavelength the non-RWA effects might be transparent even in the case of the resonant interaction. Secondly, in the semiclassical version of the model, chaotic behaviour was discovered (Zaslavsky 1981, Milonni *et al* 1983). This stimulated the interest in the fully quantised version of the model as a simple candidate for the 'quantum chaotic' system (Graham and Höhnerbach 1984, 1985). Furthermore the last problem is connected with the question of how the 'quantum non-integrability' influences the effect of collapses and revivals investigated by Eberly *et al* (1980, see also Hioe 1983 and references quoted therein). Finally the two-level atom model has its counterpart in condensed matter physics which makes it interesting from a different physical point of view (see, e.g., Reik *et al* 1982).

The studies of the Jahn-Teller effect led Judd to discover a class of exact isolated solutions (eigenstates) of the model (Judd 1979). For certain relations between the parameters of the model one can find the analytic form of the two eigenvectors of the Hamiltonian with the corresponding energy. The most complete and simple description of these solutions, also for the optical applications, has been given by Reik *et al* (1982). They observed that the isolated solutions can easily be obtained by using the Neumann series expansion for the eigenvectors in the Bargmann representation for the boson operators.

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The aim of our paper is to present some generalisations of the solutions found by Judd and rederived by Reik *et al.* Section 2 of our paper contains the brief discussion of the known solutions of the two-level atom. In the following three sections we present the new results. In particular we generalise these solutions to the cases of multilevel atomic systems (§ 3) and of autoionising systems (§ 4). Finally in § 5 we discuss analogous solutions to the problem of the interaction of a two-level system with the external field.

## 2. Two-level system

The well known Hamiltonian of the two-level system interacting with the single radiation mode has the form:

$$H = a^+ a + \omega\sigma_3 + \lambda(\sigma^+ + \sigma^-)(a^+ + a) \tag{1}$$

where $a^+$ and $a$ are the photon creation and annihilation operators, $2\omega$ is the atomic level separation frequency, $\lambda$ atom-field coupling constant and $\sigma_3$, $\sigma^+$, $\sigma^-$ are the usual spin-$\frac{1}{2}$ operators with the following commutation relations:

$$[\sigma_3, \sigma^\pm] = \pm\sigma^\pm, \qquad [\sigma^+, \sigma^-] = 2\sigma_3.$$

The frequency (energy) units are chosen in such a way that the photon frequency equals 1.

In the representation in which $\sigma_1 = \sigma^+ + \sigma^-$ is diagonal, the stationary Schrödinger equations for the two-component wavefunction $\binom{\psi_1}{\psi_2}$ take the following form:

$$\begin{aligned}(z+\lambda)(\mathrm{d}/\mathrm{d}z)\psi_1 &= (E-\lambda z)\psi_1 - \omega\psi_2 \\ (z-\lambda)(\mathrm{d}/\mathrm{d}z)\psi_2 &= (E+\lambda z)\psi_2 - \omega\psi_1.\end{aligned} \tag{2}$$

In writing the equations (2) we used the Bargmann representation for the bosonic operators $a^+ \to z$, $a \to \mathrm{d}/\mathrm{d}z$. The Hilbert-Bargmann space consists of all the analytic functions $f(z)$ for which

$$\int \mathrm{d}_2 z f^*(z) f(z) \exp(-|z|^2) < \infty. \tag{3}$$

The scalar product is defined by

$$\langle f|g\rangle = \frac{1}{\pi}\int \mathrm{d}_2 z f^*(z) g(z) \exp(-|z|^2). \tag{4}$$

In the following analysis we will follow Schweber (1967). Substituting

$$\xi = z + \lambda, \qquad \psi_1(z) = \exp(-\lambda\xi)\varphi(\xi), \qquad \varepsilon = E + \lambda^2 \tag{5}$$

we obtain a second-order equation

$$\begin{aligned}\xi(\xi - 2\lambda)\,\mathrm{d}^2\varphi/\mathrm{d}\xi^2 + [2\lambda(\varepsilon-1) + (1 - 2\varepsilon + 4\lambda^2)\xi - 2\lambda\xi^2]\,\mathrm{d}\varphi/\mathrm{d}\xi \\ + [\varepsilon^2 - \omega^2 - 4\lambda^2\varepsilon + 2\lambda\varepsilon\xi]\varphi = 0.\end{aligned} \tag{6}$$

Assuming the following series expansion for $\varphi(\xi)$:

$$\varphi(\xi) = \sum_{n=0}^{\infty} c_n \xi^n \tag{7}$$

we are led to the tridiagonal recurrence for the coefficients

$$b_n = (n-\varepsilon)c_n$$

$$b_{n+1} - \frac{1}{2\lambda}\left(\frac{n-\varepsilon+4\lambda^2}{n+1} - \frac{\omega^2}{(n+1)(n-\varepsilon)}\right)b_n + \frac{1}{(n+1)}b_{n-1} = 0. \tag{8}$$

The recurrence (8) has two linearly independent solutions $b_n^{(1)}$, $b_n^{(2)}$ with the following limit behaviour:

$$\lim_{n\to\infty}\frac{b_{n+1}^{(1)}}{b_n^{(1)}} = 0 \qquad \lim_{n\to\infty}\frac{b_{n+1}^{(2)}}{b_n^{(2)}} = \frac{1}{2\lambda}. \tag{9}$$

Because we are looking for the analytic (in the whole plane) solutions of (6) we are interested in the solution $b_n^{(1)}$. The general solution of (8) is of course a linear combination of the solutions $b_n^{(1)}$ and $b_n^{(2)}$:

$$b_n = A_1 b_n^{(1)} + A_2 b_n^{(2)}. \tag{10}$$

The values of $A_1$ and $A_2$ are determined by the initial values $b_0$ and $b_1$

$$b_0 = A_1 b_0^{(1)} + A_2 b_0^{(2)} \qquad c_1 = A_1 b_1^{(1)} + A_2 b_1^{(2)}. \tag{11}$$

The requirement $A_2 = 0$ gives:

$$\frac{b_1^{(1)}}{b_0^{(1)}} = \frac{b_1}{b_0} = \frac{1}{2\lambda}\left(2\lambda^2 - \varepsilon + \frac{\omega^2}{\varepsilon}\right). \tag{12}$$

On the other hand the proper limit behaviour (i.e. $\lim_{n\to\infty}(b_{n+1}/b_n) = 0$) is possessed by the continued fraction solution of (8)

$$\frac{b_1^{(1)}}{b_0^{(1)}} = -\cfrac{\beta_1}{\alpha_1 - \cfrac{\beta_2}{\alpha_2 - \ldots}} \tag{13}$$

where

$$\alpha_n = -\frac{1}{2\lambda}\left(\frac{n-\varepsilon+4\lambda^2}{n+1} - \frac{\omega^2}{(n+1)(n+\varepsilon)}\right)$$

$$\beta_n = \frac{1}{n+1}.$$

Combining the equations (12) and (13) we obtain a transcendental equation for the energy $\varepsilon$. This is the final formula obtained by Schweber.

By the elementary transformation of continued fractions this can be brought to the form

$$0 = B_0(\varepsilon) - \cfrac{A_1(\varepsilon)}{B_1(\varepsilon) - \cfrac{A_2(\varepsilon)}{B_2(\varepsilon) - \ddots}} \tag{14}$$

where

$$A_n(\varepsilon) = 4n\lambda^2(n-1-\varepsilon)(n-\varepsilon)$$

$$B_n(\varepsilon) = (n-\varepsilon)(n-\varepsilon+4\lambda^2) - \omega^2. \tag{15}$$

It is easy to see that $A_n(n)=0$, so $\varepsilon=n$ is the root of (14) provided the finite continued fraction

$$B_0(n)-\cfrac{A_1(n)}{B_1(n)-\cfrac{A_2(n)}{B_2(n)-\ddots-\cfrac{A_{n-1}(n)}{B_{n-1}(n)}}}$$

is equal to zero. This states a certain relation between $\omega$ and $\lambda$ (compatibility condition).

The solutions corresponding to these energies were originally found by Judd (1979) in a different context and were analysed by Reik *et al* (1982), who used a Neumann series expansion for the eigenfunctions. In this language the termination of the Neumann series determines the solutions and the compatibility conditions.

In fact probably the simplest method for obtaining the exact isolated solutions is based on the use of a very simple and natural ansatz. Because the energies of these solutions are the same as for the exactly soluble model with $\omega=0$ we look for the eigenstates formed by the finite number of excitations of the shifted photon vacuum (ground state for $\omega=0$):

$$\psi_1(z)=P_n(z)\,\mathrm{e}^{-\lambda z} \qquad \psi_2(z)=Q_{n-1}(z)\,\mathrm{e}^{-\lambda z} \tag{16}$$

where $P_n$ and $Q_{n-1}$ are the polynomials in $z$:

$$P_n(z)=p_nz^n+p_{n-1}z^{n-1}+\ldots+p_0 \tag{17}$$

$$Q_{n-1}(z)=q_{n-1}z^{n-1}+q_{n-2}z^{n-2}+\ldots+q_0. \tag{18}$$

Substituting (16)-(18) into (2) and comparing the coefficients of $z^{n+1}$ and $z^n$ in the first equation we obtain

$$\lambda p_n=\lambda p_n$$

(which is identically fulfilled) and

$$(n-\lambda^2)p_n=Ep_n \tag{19}$$

which determines the energy.

For the remaining $2n+1$ coefficients $p_k$ $(0\leqslant k\leqslant n)$ and $q_k$ $(0\leqslant k\leqslant n-1)$ we obtain $2n+1$ linear equations comparing the corresponding powers of $z$ on both sides of the equations (2). Obviously the non-trivial solution exists when the determinant vanishes, which gives the compatibility condition. In particular the first two compatibility conditions ($n=1,2$) have the explicit form

$$\omega^2+4\lambda^2=1 \qquad \text{for } n=1, \tag{20}$$

$$\omega^4+(12\lambda^2-5)\omega^2+32\lambda^4-32\lambda^2+4=0 \qquad \text{for } n=2. \tag{21}$$

For the given $n$ the compatibility condition has the form of the equation of $n$th degree in $\lambda^2$ $(\omega^2)$. Whenever $\lambda^2>0$ and $\omega^2>0$ fulfil this equation we are able to construct two independent eigenfunctions of the Hamiltonian (1). The second solution is obtained from (16) by the substitution $\psi_1'(z)=\psi_2(-z)$, $\psi_2'(z)=\psi_1(-z)$.

The number of pairs $\lambda^2$ and $\omega^2$ fulfilling the compatibility conditions increases with $n$. This can be understood if we rewrite the equations (2) in the form of an eigenproblem with respect to $\omega^2$:

$$L\psi_1=\omega^2\psi_1 \tag{22}$$

where

$$L=[(z-\lambda)\,\mathrm{d}/\mathrm{d}z-E-\lambda z][(z+\lambda)\,\mathrm{d}/\mathrm{d}z-E+\lambda z]. \tag{23}$$

It can be easily proven that if $E=n+\lambda^2$ and

$$(\mathrm{d}/\mathrm{d}z-\lambda)^n\psi_1=0 \tag{24}$$

then $(\mathrm{d}/\mathrm{d}z-\lambda)^nL\psi_1=0$, i.e. $L$ has an invariant $n$-dimensional subspace. In this language the compatibility condition is interpreted as a secular equation determining the value of $\omega^2$. Obviously the operator $L$ is not Hermitian, it can be however proved that all $\omega^2$ must be real.

Let us now focus our attention on the analytic properties of the solutions of the equations (2). From the construction our solutions (16) are certainly analytic in the whole plane. The equations (2) as a set of two first-order linear equations have two imdependent solutions. The second solution may be analytic or not. We shall check under which condition both solutions are analytic.

The set (2) has two singular points $z=\pm\lambda$. We perform the Frobenius analysis in the point $z=\lambda$, i.e. we look for the solutions in the form $\psi_i=(z-\lambda)^\rho\sum_{n=0}^{\infty}c_n^i(z-\lambda)^n$. Equating the coefficients of the same powers of $z-\lambda$ we arrive at the indicial equation

$$\rho(E+\lambda^2-\rho)=0. \tag{25}$$

Therefore both linearly independent solutions of (2) are analytic in the neighbourhood of $z=\lambda$ when $\rho$ is a non-negative integer, i.e. when the energy is equal to $n-\lambda^2$ $(n=0,1,2,\ldots)$ which is exactly the value for the isolated exact solutions. The same conditions can be obtained by analysing the analyticity in $z=-\lambda$. In fact the solution for $E=n-\lambda^2$ has its analytic counterpart which follows immediately from the symmetry considerations. Namely, as we mentioned, if $\binom{\psi_1(z)}{\psi_2(z)}$ is a solution then $\binom{\psi_2(-z)}{\psi_1(-z)}$ is also a solution. This symmetry (parity) operation applied to (16) gives a linearly independent, analytic solutions of (2). The energy level $E=n-\lambda^2$ is thus doubly degenerated†.

As a conclusion we obtain that the requirement of analyticity of both solutions in one of the singular points gives the energy for which the exact solution of (2) can be found.

The existence of exact isolated solutions does not of course allow for the general solution of the eigenvalue problem (2). Nevertheless, these solutions can serve as a test for different perturbation expansions and numerical procedures. Moreover first-order perturbation expansion around them can be obtained explicitly.

If one looks at the energy levels of (2) for example as a function of $\lambda^2$ certain regularities of the spectrum can be found. For instance the $n$th energy level with a given parity crosses $n$ times the line $E=n-\lambda^2$ if $\omega$ is not too large. Increasing $\omega$ the number of such crossings decreases, but nevertheless remains large for large $n$ and the $n$th energy level oscillates around $n-\lambda^2$. The exact isolated solution does confine the function $E_n(\lambda^2)$. One can obtain a nice interpolation formula for $E_n(\lambda^2)$ using the exact crossings of basic lines for $\lambda^2=\lambda_0^2$ (where $\lambda_0^2$ is determined by the proper compatibility condition, e.g. (20) or (21)), as well as the exact values of the derivatives of $E_n(\lambda^2)$ at those points. Such an interpolation formula works better for large $n$, since we have more exact isolated solutions then. Moreover the confinement of the energy levels to the basic lines $(E=n-\lambda^2)$ allows us to attach a definite quantum number to

† The ansatz (16) does not have a definite parity connected with the above symmetry. Therefore the second solution with the same energy exists.

each energy level. In another words we may expect that the quantum KAM theorem (Hose and Taylor 1983) should be applicable in this case and the candidate for the integrable Hamiltonian $H_0$ which approximates (1) is easy to guess: one should take the Hamiltonian (1) with $\omega$ equal to zero. The guess is based on the observation that for $\omega = 0$ the energy levels are doubly degenerated and equal to $E_n(\lambda^2) = n - \lambda^2$. In fact Graham and Höhnerbach (1985) have shown the numerical evidence for the fact that the assumptions of the QKAM theorem are fulfilled for $H$ and $H_0$.

We conclude that the relatively small difference between the non-integrable system (1) and the integrable system with $\omega = 0$ is due to the existence of the isolated exact solutions.

In the following sections we shall show that exact isolated solutions can also be found for more complicated systems. We shall use the method based on the three essential observations drawn from the presented analysis of the two-level system:

(i) the exact solutions are in fact obtained using a very simple ansatz, namely the eigenstate is a linear combination of the finite excitations of the shifted photon vacuum;

(ii) the existence of the above solutions may be attributed to the existence of the finite dimensional invariant subspace of some (in general non-Hermitian) operator;

(iii) the eigenvalues of these solutions are such that the necessary condition of analyticity of all the linearly independent solutions of the Bargmann representation equations in a singular point is fulfilled.

## 3. Multi-level systems

The Hamiltonian of the $N$-level system interacting with the single mode of the electromagnetic field has the form

$$H = \hat{\Omega} + \hat{\lambda}(a^\dagger + a) + a^\dagger a \tag{26}$$

$\hat{\Omega}$ and $\hat{\lambda}$ are here $N \times N$ Hermitian matrices; $\hat{\Omega}$ is a free atomic Hamiltonian and $\hat{\lambda}$ is a matrix of dipole transition elements. In the Bargmann representation the eigenvectors of (26) can be written as $N$ component, $z$ dependent vectors $\psi(z)$, fulfilling the Schrödinger equation:

$$[z\,\mathrm{d}/\mathrm{d}z + \hat{\lambda}(z + \mathrm{d}/\mathrm{d}z) + \hat{\Omega}]\psi(z) = E\psi(z). \tag{27}$$

It is worth stressing that for $N > 2$ the method of continued fractions is no longer suitable for solving (27).

If we use the representation in which $\hat{\lambda}$ is diagonal and assume that all eigenvalues of $\hat{\lambda}$ are different, equation (27) can be written in the form

$$(z + \lambda_i)\frac{\mathrm{d}}{\mathrm{d}z}\psi_i(z) = (E - \lambda_i z)\psi_i(z) - \sum_{j=1}^{N} \Omega_{ij}\psi_j(z). \tag{28}$$

Obviously in general the system (28) has $N$ independent solutions. The required solution should be analytic in the whole plane, in particular in all singular points of the equation (28) $z = -\lambda_i$. Assuming that $\psi$ behaves like $(z + \lambda_i)^\rho$ in the vicinity of $z = -\lambda_i$ we are led to the following indicial equation:

$$\rho^{N-1}(\rho - E - \lambda_i^2 + \Omega_{ii}) = 0. \tag{29}$$

If

$$E = n + \Omega_{ii} - \lambda_i^2 \qquad \text{for } n = 0, 1, 2, \ldots$$

then all $N$ independent solutions are analytic in $z=-\lambda_i$ In this case we may expect the existence of simple exact solutions of (28). Indeed we find this solution using the ansatz:

$$\psi(z)=\mathrm{e}^{-\lambda z}(A_n z^n+A_{n-1}z^{n-1}+\ldots+A_0) \tag{30}$$

where $A_k$ are $N$-dimensional columns. The ansatz (30) leads to the following set of equations

$$(\hat{\lambda}-\lambda)A_n=0 \tag{31a}$$

$$(\hat{\lambda}-\lambda)A_{n-1}=(E+\lambda\hat{\lambda}-\hat{\Omega}-n)A_{n-1} \tag{31b}$$

$$(\hat{\lambda}-\lambda)A_{n-k-1}=(E+\lambda\hat{\lambda}+k-\hat{\Omega}-n)A_{n-k}$$

$$-\hat{\lambda}(n-k-1)A_{n-k+1}=0 \qquad \text{for } n>k>0; \tag{31c}$$

$$(E-\hat{\Omega}+\lambda\hat{\lambda})A_0-\hat{\lambda}A_1=0. \tag{31d}$$

Equations (31) can be solved as follows:

(i) $\lambda$ has to be equal to the one of the eigenvalues of $\hat{\lambda}$, say $\lambda=\lambda_i$. $A_n$ is then proportional to the $i$th eigenvector of the finite dimensional matrix $\hat{\lambda}$;

(ii) since the matrix $(\hat{\lambda}-\lambda)$ has no inverse, equation (13$b$) can be solved if and only if

$$(E-\Omega_{ii}-n+\lambda_i^2)=0, \tag{32}$$

this condition, determining the energy $E$, is equivalent to the requirement of the analyticity of all $N$ independent solutions of (28) in the vicinity of $z=-\lambda_i$;

(iii) equation (31$b$) determines $A_{n-1}$ non-uniquely. $A_{n-1}$ may contain an arbitrary contribution parallel to $A_n$. This contribution is, however, uniquely determined from (31$c$) in the case $k=1$;

(iv) analogously $A_{n-s-1}$ is determined from (31$c$) for $k=s$ up to the projection onto $A_n$. The latter is determined also from (31$c$) but for $k=s+1$;

(v) finally equation (31$d$) plays a double role: it determines the projection of $A_0$ onto $A_n$ and states $N-1$ compatibility conditions.

For $n=0$ the compatibility condition requires that $\hat{\Omega}$ and $\hat{\lambda}$ have a common eigenvector. Here we present a non-trivial result for $n=1$

$$E=1+\Omega_{ii}-\lambda_i^2 \tag{33}$$

and the compatibility condition reads

$$\sum_{k\neq i}\frac{\Omega_{jk}\Omega_{ki}}{\lambda_k-\lambda_i}-\Omega_{ji}\left(\lambda_i-\sum_{k\neq i}\frac{|\Omega_{ik}|^2}{\lambda_k-\lambda_i}+\frac{1+\Omega_{ii}+\lambda_i(\lambda_j-\lambda_i)}{(\lambda_j-\lambda_i)}\right)=0. \tag{34}$$

In order to get more insight into the obtained result we shall analyse in the remainder of this section an example of a three-level system. We shall assume the particular form of the matrices $\hat{\Omega}$ and $\hat{\lambda}$

$$\hat{\Omega}=\begin{pmatrix}\omega_1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \omega_2\end{pmatrix}, \qquad \hat{\lambda}=\begin{pmatrix}0 & \lambda_1 & 0\\ \lambda_1 & 0 & \lambda_2\\ 0 & \lambda_2 & 0\end{pmatrix}. \tag{35}$$

The energy level of the free atom are $\omega_1$, 0, $\omega_2$, while the dipole transition matrix elements $\lambda_1$, $\lambda_2$. Direct transitions from the lowest to the highest atomic state is forbidden. In the first step we diagonalised $\hat{\lambda}$

$$\hat{\lambda} = \begin{pmatrix} \lambda_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\lambda_0 \end{pmatrix} \tag{36}$$

where $\lambda_0 = (\lambda_1^2 + \lambda_2^2)^{1/2}$. In this representation

$$\hat{\Omega} = \begin{pmatrix} \Omega_1 & \Delta & -\Omega_1 \\ \Delta & \Omega_2 & -\Delta \\ -\Omega_1 & -\Delta & \Omega_1 \end{pmatrix} \tag{37}$$

where

$$\Omega_1 = \tfrac{1}{2}\left(\frac{\omega_1\lambda_1^2 + \omega_2\lambda_2^2}{(\lambda_1^2+\lambda_2^2)}\right) \qquad \Omega_2 = \frac{\omega_1\lambda_2^2 + \omega_2\lambda_1^2}{\lambda_1^2+\lambda_2^2} \tag{38}$$

$$\Delta = \frac{1}{\sqrt{2}}\frac{\lambda_1\lambda_2}{\lambda_1^2+\lambda_2^2}(\omega_1 - \omega_2).$$

Equation (28) has now three singular points $z = 0, \pm\lambda_0$. For $N = 1$ the corresponding energies are

$$E = 1 + \Omega_1 - \lambda_0^2 \qquad \text{analyticity in } z = \pm\lambda_0, \tag{39a}$$

$$E = 1 + \Omega_2 \qquad \text{analyticity in } z = 0. \tag{39b}$$

We should be therefore be able to find two doubly degenerated solutions in the case (39*a*). This follows independently from the fact that if

$$\psi = \begin{pmatrix} \psi_1(z) \\ \psi_2(z) \\ \psi_3(z) \end{pmatrix}$$

is a solution, then

$$\begin{pmatrix} \psi_3(-z) \\ \psi_2(-z) \\ \psi_1(-z) \end{pmatrix}$$

is also the solution. In fact we find that the ansatz (30) with $n = 1$, $\lambda = \pm\lambda_0$ works provided

$$A_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad A_0 = \pm\frac{1}{\lambda_0}\begin{pmatrix} \lambda_0^2 + \Delta^2 + \Omega_1^2 \\ \Delta \\ -\tfrac{1}{2}\Omega_1 \end{pmatrix}$$

and the two compatibility conditions hold:

$$\begin{aligned} &(1 + \tfrac{1}{2}\Omega_1 - 2\lambda_0^2 - \Delta^2 - \Omega_1^2 - \Omega_2)\Delta/\lambda_0 = 0 \\ &(\Omega_1/2\lambda_0)(4\lambda_0^2 + 2\Delta^2 + \Omega_1^2 - 1) + \Delta^2/\lambda_0 = 0. \end{aligned} \tag{40}$$

The equations (40) have non-trivial solutions (for $(\Omega_1/(1+\Omega_1))(1-\Omega_1^2-4\lambda_0^2)>0$). Note that knowing the two independent solutions of (28) we are able in principle to find the third independent one. In general however it will be non-analytic at $z=0$, unless incidentally $n+\Omega_1-\lambda_0^2=k+\Omega_2$ for some natural numbers $n, k$.

Similarly the ansatz (30) allows us to find the solution with $\lambda=0$ if

$$A_1=\begin{pmatrix}0\\1\\0\end{pmatrix} \quad \text{and} \quad A_0=\frac{1}{\lambda_0}\begin{pmatrix}-\Delta\\0\\-\Delta\end{pmatrix},$$

where the energy is given by (39*b*) and the compatibility condition reduces to a single equation

$$(1+\Omega_2)\Delta=0. \tag{41}$$

We should stress that however the number of compatibility conditions increases with the dimension of the atomic Hilbert space we are still able to find a large class of exact solutions. In the discussed example the compatibility conditions span at least a two-dimensional manifold in the four-dimensional space of $(\lambda, \Omega_1, \Omega_2, \Delta)$.

Once more the existence of isolated exact solutions, apart from possible specific applications, may be helpful in understanding the general features of the non-integrable systems (QKAM theorem, regularity of the spectrum, etc).

## 4. Autoionising atom

Up to now we were able to generalise Juddian solutions to the case of an $N$-level system. A natural question arises whether it can be done also for infinitely dimensional atomic Hamiltonians. In this section we shall show that it is indeed possible. To this end we shall discuss a model of an autoionising atom, which was recently widely investigated in the literature (Lambropoulos and Zoller 1981, Agarwall *et al* 1984, Rzazewski and Eberly 1981). The simplest model of the field induced autoionisation consists of an electron which may occupy a ground state, a bound excited state and a continuum of ionised states. The excited state is coupled to the continuum via a static configuration mixing potential. The ground state couples to the continuum and to the excited state through electric dipole transitions. There are two possible ionisation channels: direct and indirect (i.e. through the excited bound state). The probability amplitudes of the decay in these channels interfere and may in principle cancel each other. This happens in fact for the particular value of the coupling electric field. At the same time the electron undergoes coherent Rabi oscillations between the ground and the excited state. Such an effect termed 'confluence of coherences' (Rzążewski and Eberly 1981) leads in practice to a population trapping† in the ground state (Coleman and Knight 1982) as well as to a remarkable narrowing of spectral lines. In the mathematical sense it corresponds to an appearance of an exact bound state of the full Hamiltonian.

After a partial diagonalisation of a part of the system (excited state–continuum, see Fano 1961), the free atomic Hamiltonian consists of the ground state and a new continuum. The whole information about an autoionising resonance is contained then

† Of course in realistic models trapping is never perfect (Agarwal *et al* 1984).

in a new dipole coupling strength between the ground state and the new continuum. The Hamiltonian therefore has the form:

$$H = a^\dagger a + \int_{-\infty}^{\infty} [\mathrm{d}^*(\omega)|\omega\rangle\langle 0| + \mathrm{d}(\omega)|0\rangle\langle\omega|](a^\dagger + a) + \int_{-\infty}^{\infty} \omega|\omega\rangle\langle\omega|\,\mathrm{d}\omega \tag{42}$$

$|0\rangle$, $|\omega\rangle$ denote the atomic ground and continuum states (with energies $0, \omega$). The function

$$\mathrm{d}(\omega) = \frac{d}{(4\pi\Gamma)^{1/2}}\left(\frac{\Gamma}{\omega - \omega_0 - \mathrm{i}\Gamma} + \frac{1}{(1-\mathrm{i}q)}\frac{\Gamma_1}{(\omega - \omega_0 + \mathrm{i}\Gamma_1)}\right) \tag{43}$$

determines $\omega$ dependent transition matrix elements from $|0\rangle$ to $|\omega\rangle$. $\Gamma$ is here an autoionisation width, $\omega_0$ is a frequency of the auotionising resonance (we assume that $\omega_0$ is not very different from a photon frequency $\omega = 1$). Finally $q$ is a Fano asymmetry parameter, which measures a relative strength of the two mentioned ionisation channels, $\Gamma_1$ is a width of the background (typically $\Gamma_1 \gg \Gamma, |\omega_0 - 1|$). The integrals over $\omega$ are evaluated from $-\infty$ to $\infty$, which means that we neglect the threshold effects.

This approximation simplifies the model essentially, but is in fact well justified only in the case when the autoionising resonance is sufficiently far from the threshold. Unfortunately non-RWA terms involve multiphoton effects and thus are usually affected by the thresholds.

The rotating wave approximation version of (43) has a constant of motion (excitation number)

$$N = a^\dagger a + \int |\omega\rangle\langle\omega|\,\mathrm{d}\omega.$$

For each definite $N$ one finds the population trapping in the ground state if the parameters $d$, $\Gamma$, $\omega_0$, $q$ fulfil some $N$ dependent relation. Since this relation cannot be fulfilled for every $N$ simultaneously, the effect of confluence of coherences vanishes or at least decreases significantly due to eventual fluctuations of $N$.

The inclusion of the counter-rotating terms in (42) should destroy the confluence, since $N$ is no more a constant and there are more possible channels of decay due to multiphoton processes. Typically, however, if $\omega_0$ is not far from resonance ($\omega_0 = 1$) and $\Gamma$, $d \ll 1$, the decay rate of the ground state population may be very small, i.e. of the order $\Gamma((\omega_0 - 1)/\omega_0)$ or $\Gamma^2/\omega_0$.

Nevertheless, we cannot exclude the fact that, even in the case without RWA, for some arrangement of the atomic parameters the amplitudes of the different decay channels interfere destructively leading to a perfect or more probably approximate population trapping. Such an effect would have a slightly different physical meaning to the confluence discussed and should rather be termed confluence of many coherences. The question of whether this effect is present or not is of general importance in the theory of decaying systems. In fact this question motivated us to look for isolated exact eigenmodes of the Hamiltonian (42).

The model (42) resembles in many respect a two-level system. To see this we shall eliminate the continuum using a method similar to the one used by Rzążewski (1983) and Lewenstein *et al* (1984). The time dependent wavefunction of (42) has a form

$$\psi(t) = \alpha(t)|0\rangle + \int \beta(\omega, t)|\omega\rangle\,\mathrm{d}\omega \tag{44}$$

where $\alpha$ and $\beta$ are vectors in the Hilbert space of the field oscillator. The Schrödinger equation reads

$$\dot{\alpha}(t) = -\mathrm{i}(a^{\dagger} + a) \int \mathrm{d}(\omega)\, \beta(\omega, t)\, \mathrm{d}\omega - \mathrm{i}a^{\dagger}a\alpha(t) \tag{45a}$$

$$\dot{\beta}(\omega, t) = -\mathrm{i}\omega\beta(\omega, t) - \mathrm{i}a^{\dagger}a\beta(\omega, t) - \mathrm{i}\,\mathrm{d}^{*}(\omega)(a^{\dagger} + a)\alpha(t). \tag{45b}$$

Now we solve equation (45*a*) and eliminate $\beta(\omega, t)$ in favour of an integrated amplitude $\beta(t) = \int \mathrm{d}(\omega)\, \beta(\omega, t)\, \mathrm{d}\omega$. This leads to the following (non-Hermitian) equations for $\beta$ and $\alpha$:

$$\dot{\alpha} = -\mathrm{i}(a^{\dagger} + a)\beta - \mathrm{i}a^{\dagger}a\alpha \tag{46a}$$

$$\dot{\beta} = -(\Gamma + \mathrm{i}\omega_0 + \mathrm{i}a^{\dagger}a)\alpha - \mathrm{i}\kappa M(a^{\dagger} + a)\alpha - \kappa(a^{\dagger} + a)^2\beta - \kappa(a - a^{\dagger})\alpha + K(t). \tag{46b}$$

The inhomogenity $K(t)$ depends only on the initial value $\beta(\omega, t = 0)$ and tends to zero as $t \to \infty$. We shall neglect it in the following. The constants are defined

$$\kappa = d^2/4(1 + q^2)\Gamma \tag{47}$$

$$M = \Gamma q^2 + \mathrm{i}(\omega_0 - 2\Gamma q). \tag{48}$$

The eigensolutions of the homogeneous part of (46) have a simple time dependence $\alpha(t) = \mathrm{e}^{-\mathrm{i}Et}\alpha$, $\beta(t) = \mathrm{e}^{-\mathrm{i}Et}\beta$, where $E$ is in general a complex number with a negative imaginary part. Vanishing of $\mathrm{Im}(E)$ would correspond to the 'confluence of many coherences'. Using the Bargmann representation for the field mode, we obtain

$$E_{\alpha} = z(\mathrm{d}/\mathrm{d}z)\alpha + (z + \mathrm{d}/\mathrm{d}z)\beta \tag{49a}$$

$$E_{\beta} = z(\mathrm{d}/\mathrm{d}z)\beta + [\omega_0 - \mathrm{i}\Gamma - \mathrm{i}\kappa(z + \mathrm{d}/\mathrm{d}z)^2]\beta + \kappa(M + \mathrm{i})z\alpha + \kappa(M - \mathrm{i})(\mathrm{d}/\mathrm{d}z)\alpha. \tag{49b}$$

Note the appearance of the qualitatively new term $(z + \mathrm{d}/\mathrm{d}z)^2$ on the RHS of (49*b*). We must discuss the case $q = \infty$ separately (symmetric Fano resonance), since in this case

$$\kappa \to 0, \qquad \kappa M \to \tfrac{1}{4}d^2. \tag{50}$$

In this limit the equations (49) are equivalent to the equations (2) if we substitute

$$2\omega \to (\omega_0 - \mathrm{i}\Gamma), \qquad \lambda^2 \to \tfrac{1}{4}d^2. \tag{51}$$

The analysis from § 2 can, therefore, be applied and we find eigenmodes of the form

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \exp(\pm\tfrac{1}{2}dz)\left(\begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} z^n + \ldots + \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix}\right) \tag{52}$$

with the energies

$$E = n - \tfrac{1}{4}d^2 - \mathrm{i}\tfrac{1}{2}\Gamma. \tag{53}$$

The compatibility conditions (20) and (21) and others can, however, be fulfilled only for $(\omega_0 - \mathrm{i}\Gamma)$ real. This implies either $\Gamma = 0$ (which is exactly the pure two-level atom case) or $\omega_0 = 0$. In the latter case the equation for $n = 1$ takes the form:

$$\tfrac{1}{4}d^2 - \Gamma^2 - 1 = 0. \tag{54}$$

At first sight the solutions with $\omega_0 = 0$ are not very interesting. What is physically interesting is the case of the resonances, i.e. $\omega_0 = 2k + 1$, where $k$ is natural. Equations (49) have, however, an alternative interpretation. We can interpret $(a + a^{\dagger})$ as a slowly

varying, fluctuating amplitude of the EM field. Equations (49) arise as a result of the RWA approximation with respect to the fast oscillations of the phase of the field (in this case $|0\rangle$ should have some negative energy, comparable with the phase oscillations frequency). In this language the case $\omega_0 = 0$ corresponds to the exact resonance and is indeed physically relevant.

In the case $q \neq \infty$ we first analyse the analytical properties of the solutions of (49). Due to the appearance of the second derivatives in (49*b*) the only singular point is $z = 0$. The requirement of analyticity of $\alpha$, $\beta$ and $\mathrm{d}\beta/\mathrm{d}z$ in $z = 0$ for all solutions of (49) gives

$$E = n + 1 - \mathrm{i}M \qquad n = 0, 1, \ldots . \tag{55}$$

Note that the energy width $\mathrm{Im}(E) = -\Gamma q^2$ vanishes only for the somewhat peculiar case $q = 0$, and does not lead to any kind of unusual narrowing. We use once more the ansatz (30) and write

$$\alpha = \sum_{k=0}^{n} A_k z^k \qquad \beta = \sum_{k=0}^{n-1} B_k z^k. \tag{56}$$

This way we obtain the following compatibility conditions

$$(\mathrm{i}M - 2 - \mathrm{i}\Gamma + \omega_0 - \mathrm{i}\kappa)(\mathrm{i}M - 1) = \kappa(M - \mathrm{i}) \qquad \text{for } n = 1 \tag{57}$$

and

$$(\mathrm{i}M - 2 - \mathrm{i}\Gamma + \omega_0 - 3\mathrm{i}\kappa)(\mathrm{i}M - 1)(\mathrm{i}M - 3) = \kappa(3\mathrm{i}M^2 - 6M - 7\mathrm{i}) \qquad \text{for } n = 2. \tag{58}$$

Each of the complex number equations implies in fact two real number relations between $\Gamma$, $q$, $\omega_0$ and $\frac{1}{4}d^2$.

Up to now all the exact solutions, which we have discussed could be found under the corresponding analyticity condition which determined the energy. The slight modification of the ansatz (30) allows, however, for finding a class of exact solutions of (49) which do not fulfil the analyticity requirement. This is connected with the appearance of the term $(z + \mathrm{d}/\mathrm{d}z)^2$ on the RHS of (49*b*). The new solutions are obtained via the Gaussian transformation

$$\alpha = \exp(\tfrac{1}{2}\lambda z^2) \qquad \beta = \exp(\tfrac{1}{2}\lambda z^2)v. \tag{59}$$

$\lambda$ is chosen in such a way that the $z^2$ term in (49*b*) is cancelled. Using (59) we obtain

$$z\frac{\mathrm{d}}{\mathrm{d}z}u + [z(1+\lambda) + \mathrm{d}/\mathrm{d}z]v + \lambda z^2 u = Eu \tag{60a}$$

$$[1 - 2\mathrm{i}\kappa(1+\lambda)]z(\mathrm{d}/\mathrm{d}z)v + (\omega_0 - \mathrm{i}\Gamma)v - \mathrm{i}\kappa(\lambda+1)v + \kappa[(M+\mathrm{i})z + (M-\mathrm{i})\lambda z]u + \kappa(M-\mathrm{i})(\mathrm{d}/\mathrm{d}z)u = Ev. \tag{60b}$$

Transformation (59) does not change the analytic properties of the solutions in $z = 0$. $\lambda$ is defined as follows

$$\lambda = \mathrm{i}\kappa(\lambda + 1)^2. \tag{61}$$

Additionally normalisability of the solutions requires $|\lambda| < 1$. This inequality can be fulfilled by one of the two roots of (61). The exact solutions of (60) are then obtained substituting

$$u = \sum_{k=0}^{n-1} u_k z^k \qquad v = \sum_{k=0}^{n} v_k z^k. \tag{62}$$

The general formula for the energy reads

$$E = n - 1 + \omega_0 - i\Gamma + iM - i\kappa(\lambda + 1)(2n - 1). \tag{63}$$

The cases $n = 0$ and $n = 1$ do not allow for non-trivial solutions of the corresponding compatibility conditions. The first non-trivial solution is obtained for $n = 2$ if

$$1 + iM - 2i\kappa(\lambda + 1) = 0. \tag{64}$$

From (64) we immediately obtain

$$\mathrm{Im}(E) = -\Gamma(1 + \tfrac{1}{2}q^2)$$

and the solution found does not show any extraordinary narrowing or population trapping effects. We cannot, however, exclude the possibility of these effects for the higher order solutions.

## 5. Interaction with external fields

We have been able to find a class of exact solutions for non-integrable quantum systems, using the simple ansatz (30) or (59). The essential point of our method was that after extracting an exponential factor from the wavefunction the resulting equations may have polynomial solutions.

However, it remains unclear whether the solutions obtained do have some analogues in the semiclassical versions of the models discussed. There are, in principle, two stages of the semiclassical limit:

(a) quantum atoms in an external time dependent field;

(b) classical spins interacting with the classical oscillator (Milonni *et al* 1983).

We do not expect to obtain results in the case (b) (i.e. to find exact classical tori). The spins considered here are $\frac{1}{2}$ or 1 and are rather far from the classical limit.

We have been, however, able to construct the analogue of the ansatz (30) in case (a). The only non-trivial results were obtained for the case of an autoionising atom with $q = \infty$. The time dependent Schrödinger equation reads in this case:

$$\begin{aligned} \dot{\alpha} &= if(t)\beta \\ \dot{\beta} + (\Gamma + i\omega_0)\beta &= -if(t)\alpha. \end{aligned} \tag{65}$$

Once more we interpret these equations rather as the RWA equations with a slowly varying field amplitude $f(t)$. In this case $\omega_0$ has the meaning of the detuning of the fast laser frequency from the transition frequency between the ground state and the Fano resonance. We consider

$$f(t) = \lambda_0 + \lambda_1 \cos(t)$$

i.e. the field amplitude is modulated with frequency 1. Obviously such a model has a direct experimental relevance.

We were able to find a non-trivial solution of (65) using the ansatz

$$\begin{aligned} \alpha(t) &= \exp(-iEt + a_1 \cos(t) + a_2 \sin(t))P_n(\cos(t), \sin(t)) \\ \beta(t) &= \exp(-iEt + a_1 \cos(t) + a_2 \sin(t))R_n(\cos(t), \sin(t)) \end{aligned} \tag{66}$$

$P_n$ and $R_n$ are $n$th degree polynomials of the variables $\cos(t)$, $\sin(t)$. An exponential pre-factor should enable us to use (66). In fact for $n = 1$ we find two pairs of the solutions of the form (66) provided $a_2 = 0$, $a_1 = \pm\lambda_1$.

The compatibility conditions are

$$\lambda_0 = \pm\tfrac{1}{2}, \qquad \omega_0 = 0, \tfrac{1}{4}\Gamma^2 = \lambda_1^2$$

and

$$E = \mp\tfrac{1}{2} - i\tfrac{1}{2}\Gamma$$

This corresponds to an autoionising atom on resonance.

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